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$$7 - 14h^2 + 7h^4 - h^6 = 0.$$

This same equation can also be obtained by taking $\theta = 2\pi/7$ and $\pi - \theta = 5\pi/7$ or by taking $\theta = \pi/7$ and $\pi - \theta = 6\pi/7$ and proceeding in a manner similar to the above.

We may find the *Nonagon Cubic* by the same process. For, let n be a side of a regular nonagon inscribed in a circle whose radius is unity. Then $n = 2 \sin \pi/9$. Also $\sin \pi/3 = \sin (\pi - \pi/3) = \sin 2\pi/3 = 2 \sin \pi/3 \cos \pi/3$. Hence, dividing throughout by $\sin \pi/3$, we have

$$1 = 2 \cos \pi/3 = 2(1 - 4 \sin^2 \pi/9)\sqrt{(1 - \sin^2 \pi/9)}.$$

Square both members of this equation, unite similar terms, and we have

$$3 - 36 \sin^2 \pi/9 + 96 \sin^4 \pi/9 - 64 \sin^6 \pi/9 = 0.$$

Substitute for $\sin \pi/9$ its value $n/2$, and we have for the *Nonagon Cubic*

$$3 - 9n^2 + 6n^4 - n^6 = 0.$$

Hence, by Horner's Method, we have

$$n^2 = .4679111137620439295952l5,$$

and finally

$$n = .68404028665133746608819 9.$$

The *Undecagon Quintic* can be found in the same way. For let u be a side of a regular undecagon inscribed in a circle whose radius is unity. Then $u = 2 \sin \pi/11$.

Also $\sin 5\pi/11 = \sin (\pi - 5\pi/11) = \sin 6\pi/11$. Proceeding as before we find for the *Undecagon Quintic*

$$11 - 55u^2 + 77u^4 - 44u^6 + 11u^8 - u^{10} = 0.$$

Hence, by Horner's Method, we have

$$u^2 = .3174929343376376622763767,$$

and finally

$$u = .5634651136828593954228358.$$

REPLIES TO QUESTIONS.

17. In analytic geometry, simplicity and directness are gained by making the condition for the collinearity of three points and the equation of the straight line depend upon the determinant formula for the area of a triangle. Similar advantages are gained by making the condition of complanarity of four points and the equation of the plane depend upon the determinant formula for the volume of a tetrahedron. The former is given in the texts. Why should not the latter be given? A uniform method of developing these two determinants is desired from some contributor.

REPLY BY A. M. KENYON, Purdue University.

Let ABC be a triangle whose vertices are (x_1, y_1), (x_2, y_2), and (x_3, y_3) and let A', B', C', be the projections on the x-axis of these points. We may suppose that the notation has been so chosen that $x_1 \leq x_2 \leq x_3$; and we assume at first that the vertices of the triangle all lie in the upper half plane.

Area ABC =[1] $\pm$ [Area $A'ACC'$ − Area $A'ABB'$ − Area $B'BCC'$]

$$= \pm \tfrac{1}{2}[(y_1 + y_3)(x_3 - x_1) - (y_1 + y_2)(x_2 - x_1) - (y_2 + y_3)(x_3 - x_2)]$$

or, setting $y_1 + y_2 + y_3 = s$,

$$= \pm\frac{1}{2}\left[-(s-y_2)\begin{vmatrix} x_1 & 1 \\ x_3 & 1 \end{vmatrix} + (s-y_3)\begin{vmatrix} x_1 & 1 \\ x_2 & 1 \end{vmatrix} + (s-y_1)\begin{vmatrix} x_2 & 1 \\ x_3 & 1 \end{vmatrix}\right]$$

$$= \pm\frac{1}{2}\begin{vmatrix} s-y_1 & x_1 & 1 \\ s-y_2 & x_2 & 1 \\ s-y_3 & x_3 & 1 \end{vmatrix} = \pm\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

If the triangle does not lie wholly in the upper half plane, we may choose a new x-axis, at a distance a below the old one such that this condition is satisfied; then,

$$\text{Area} = \pm\frac{1}{2}\begin{vmatrix} x_1 & y_1+a & 1 \\ x_2 & y_2+a & 1 \\ x_3 & y_3+a & 1 \end{vmatrix} = \pm\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

The symmetry of the result shows that it is independent of the order of choice of the vertices except as to the sign factor. The + sign holds when A, B, C, is the positive direction of the perimeter of the triangle; i. e., such that if one traversed the perimeter passing through the vertices in the order A, B, C, the area inside the triangle would lie always on the left.

Let $ABCD$ be a tetrahedron whose vertices are (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), and (x_4, y_4, z_4); and let A', B', C', D', be their projections on the xy-plane. We may suppose the notation to have been so chosen that A', B', C', is the positive direction of the perimeter of the triangle $A'B'C'$, and that D' lies within the angle $A'B'C'$. We assume that the vertices of the tetrahedron all lie on the positive side of the xy-plane. Then,

Vol. $ABCD$ =[2]± [Vol. $A'B'C'ABC$ − Vol. $A'B'D'ABD$
− Vol. $B'C'D'BCD$[3]± Vol. $C'A'D'CAD$].

By means of the rule for finding the volume of a truncated triangular prism, this reduces, on setting $z_1 + z_2 + z_3 + z_4 = s$, as follows:

$$\text{Vol. } A'B'C'ABC = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\frac{s-z_4}{6}, \quad \text{Vol. } A'B'D'ABD = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_4 & y_4 & 1 \end{vmatrix}\frac{s-z_3}{6}.$$

$$\text{Vol. } B'C'D'BCD = \begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix}\frac{s-z_1}{6}, \quad \text{Vol. } C'A'D'CAD = \pm\begin{vmatrix} x_3 & y_3 & 1 \\ x_1 & y_1 & 1 \\ x_4 & y_4 & 1 \end{vmatrix}\frac{S-z_2}{6};$$

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[1] The + sign holds when B and B' are on the same side of AC, and the − sign when they are on opposite sides.

[2] The + sign holds when D and D' are on the same side of the plane ABC, and the − sign when they are on opposite sides.

[3] The + sign holds when D' is outside the triangle $A'B'C'$, and the − sign when it is inside.

where in the last, the + or − sign is to be taken according as D' is inside or outside the triangle $A'B'C'$. Substituting these values above,

$$\text{Vol. } ABCD = \pm\frac{1}{6}\begin{vmatrix} s-z_1 & x_1 & y_1 & 1 \\ s-z_2 & x_2 & y_2 & 1 \\ s-z_3 & x_3 & y_3 & 1 \\ s-z_4 & x_4 & y_4 & 1 \end{vmatrix} = \pm\frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}.$$

If the tetrahedron is not wholly above the xy-plane, we may choose a new xy-plane below the old one such that this condition is satisfied and, as in the case of the triangle, show that the formula still holds.

The result is again independent of the order of choice of the vertices except as to sign which is affected by a change of order in an obvious manner.

14. In the process of solving a certain physical problem Professor H. S. Uhler, of Yale University, was led to the definite integral

$$\int_0^a (a^2 - x^2)x dx \int_{a-x}^{a+x} \frac{e^{-cy}}{y} dy,$$

for which he found the value

$$\frac{1}{c^2}\left[a^2 - \frac{3a}{c} + \frac{1}{c}\left(a + \frac{3}{2c}\right)(1 - e^{-2ac})\right],$$

a and c being positive constants. Professor Uhler would like to see how other persons attack the problem of evaluating this integral.

(Note: Several good solutions have been received in reply to the above request one of which was published in December and others may be published later; but since there are some who would be interested to see Professor Uhler's method we give it below.—Editor.)

Reply by H. S. Uhler, Yale University.

Let

$$I \equiv \int_0^a (a^2 - x^2)x dx \int_{a-x}^{a+x} \frac{e^{-cy}}{y} dy.$$

Differentiate I with respect to the parameter c, then

$$\frac{dI}{dc} = -\int_0^a (a^2 - x^2)x dx \int_{a-x}^{a+x} e^{-cy} dy,$$

which can be evaluated at once and gives

$$\frac{dI}{dc} = \frac{2a^2}{c^3} - \frac{6a}{c^4} + \frac{6}{c^5} - \left(\frac{6a}{c^4} + \frac{6}{c^5} + \frac{2a^2}{c^3}\right)e^{-2ac}.$$

We can now integrate both sides of this equation, thus

$$\int_0^I dI = \int_c^\infty \left\{\frac{2a^2}{c^3} - \frac{6a}{c^4} + \frac{6}{c^5} - \left(\frac{6a}{c^4} + \frac{6}{c^5} + \frac{2a^2}{c^3}\right)e^{-2ac}\right\} dc$$

whence

$$I = \frac{1}{c^2}\left[a^2 - \frac{3a}{c} + \frac{1}{c}\left(a + \frac{3}{2c}\right)(1 - e^{-2ac})\right].$$